Convergence Tests

Academic Resource Center

Series

- Given a sequence $\{a_0, a_1, a_2, \ldots, a_n\}$
- The sum of the series, $S_n =$

$$\sum_{k=1}^{n} a_k$$

- A series is convergent if, as n gets larger and larger, S_n goes to some finite number.
- If S_n does not converge, and S_n goes to ∞, then the series is said to be divergent

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Geometric and P-Series

- The two series that are the easiest to test are geometric series and p-series.
- Geometric is generally in the form
- P-series is generally in the form

$$\sum_{k=1}^{\infty} ar^k$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

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Geometric Series

- A geometric series is a series in which there is a constant ratio between successive terms

- 1 +2 + 4 + 8 + … each successive term is the previous term multiplied by 2

- each successive term is the previous term squared.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{256}+\dots$$

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Geometric Series

- S_n= $\sum_{k=1}^{\infty} ar^k$ = ar +ar^2 + ar^3 + … +ar^k

- As a result, if |r|<1, the geometric series will converge to , and if |r| 1 the series will diverge.

$$\frac{a}{1-r} \qquad \geq$$

P-Series

- Given a series $\sum_{n=1}^{\infty} \frac{1}{n^p}$
- This series is said to be convergent if p>1,
- And divergent if p 1

$\leq$

Geometric and P-Series Examples

$$\sum_{n=1}^{\infty} \frac{3^n}{4^{n-1}} = \sum_{n=1}^{\infty} 3\left(\frac{3}{4}\right)^{n-1}$$

This series is geometric with a=3 and r =3/4. Since r<1, this series will converge.

The Sum of the series, S

$$S = \frac{a}{1-r}$$

So S= 3/(1-3/4) = 12

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

Here, p=3, so p>1. Therefore our series will converge

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{2}}$$

Here, p=1/2, so p<1. Therefore our series will diverge

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Convergence Tests

- Divergence test
- Comparison Test
- Limit Comparison Test
- Ratio Test
- Root Test
- Integral Test
- Alternating Series Test

Divergence Test

- Say you have some series $\sum_{n=0}^{\infty} a_n$
- The easiest way to see if a series diverges is this test
- Evaluate L= $\lim_{n \to \infty} a_n$
- If L $\neq$ 0, the series diverges
- If L=0, then this test is inconclusive

Divergence Test Example

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$$

Let's look at the limit of the series

$$\lim_{n \to \infty} \frac{n^2}{5n^2+4} = \lim_{n \to \infty} \frac{n^2}{5n^2} = \frac{1}{5} \neq 0$$

Therefore, this series is divergent

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n \to \infty} \frac{1}{n^2} = 0$$

The limit here is equal to zero, so this test is inconclusive. However, we should see that this a p-series with p>1, therefore this will converge.

Comparison Test

- Often easiest to compare geometric and p series.

- Let and be series with non-negative terms.
- If $\sum a_k$ for k, $\sum b_k$ as k gets big, then
  - If converges, then converges
  - If diverges, then diverges

$$b_k > a_k$$

$\sum b_k$ $\sum a_k$

$\sum b_k$ $\sum a_k$

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Comparison Test Example

$$\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$$

Test to see if this series converges using the comparison test

This is very similar to $\sum_{n=1}^{\infty} \frac{1}{3^n}$ which is a geometric series so it will converge

And since $\sum_{n=1}^{\infty} \frac{1}{3^n} > \sum_{n=1}^{\infty} \frac{1}{3^n - 1}$ our original series will also converge

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Limit Comparison Test

- Let $\sum a_k$ and $\sum b_k$ be series with non-negative terms.
- Evaluate $\lim_{k \to \infty} \frac{a_k}{b_k}$
- If lim=L, some finite number, then both $\sum a_k$ and $\sum b_k$ either converge or diverge.
- $\sum a_k$ and $\sum b_k$ are generally geometric series or p-series, so seeing whether these series are convergent is fast.

Limit Comparison Test Example

$$\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$$

Determine whether this series converges or not

Compare it with $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ so $\displaystyle\lim_{n \to \infty} \frac{\frac{9^n}{3+10^n}}{\left(\frac{9}{10}\right)^n} = 1 > 0$

And since $\displaystyle\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ Is a geometric series with r<1, this series converges, therefore so does our original series

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Ratio Test

- Let $\sum a_k$ be a series with non-negative terms.
- Evaluate L= Lim $\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$
  - If L < 1, then $\sum a_k$ converges
  - If L > 1, then $\sum a_k$ diverges
  - If L = 1, then this test is inconclusive

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Ratio Test Example

Test for convergence

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$

Look at the limit of $\left|\frac{a_{n+1}}{a_n}\right|$

$$\lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right| = \lim_{n \to \infty} \frac{(n+1)^3}{3^{n+1}} \bullet \frac{3^n}{n^3}$$

$$= \frac{1}{3} \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^3 = \frac{1}{3} \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^3 = \frac{1}{3} < 1$$

Since L<1, this series will converge based on the ratio test

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Root Test

- Let $\sum a_n$ be a series with non-negative terms.
- Useful if a_n involves nth powers
- Evaluate L= $\lim_{n \to \infty} (a_n)^{\frac{1}{n}}$
  - If L < 1, $\sum a_n$ is convergent
  - If L > 1, $\sum a_n$ is divergent
  - If L = 1, then this test is inconclusive

Root Test Example

Test for convergence

$$\sum_{n=1}^{\infty}\left(\frac{4n+5}{5n+6}\right)^{n}$$

Lets evaluate the limit, L =$\lim_{n\to\infty}(a_n)^{\frac{1}{n}}$

$$\lim_{n\to\infty}\left(\left(\frac{4n+5}{5n+6}\right)^{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{4n+5}{5n+6} = \frac{4}{5} < 1$$

By the root test, since L<1, our series will converge.

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Integral Test

- Given the series $\sum a_k$, let a_k = f(k)
- f must be continuous, positive, and decreasing for x > 0
- $\sum_{k=0}^{\infty} a_k$ will converge only if $\int_0^{\infty} f(x)\,dx$ converges.
- If $\int_0^{\infty} f(x)\,dx$ diverges, then the series will also diverge.
- In general, however,

$$\sum_{k=1}^{\infty} a_k \neq \int_1^{\infty} f(x)\,dx$$

Integral Test Example

Test for convergence

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$$

So let $f(x) = \frac{1}{(2x+1)^3}$

Since x>0, f(x) is continuous and positive.
f'(x) is negative so we know f(x) is decreasing.

Now let's look at the integral

$$\int_1^{\infty} \frac{1}{(2x+1)^3} dx = -\frac{1}{2} \bullet 2 \lim_{t \to \infty} \left[\frac{1}{(2x+1)^2}\right]_1^t$$

$$= -\lim_{t \to \infty} \left(\frac{1}{(2t+1)^2} - \frac{1}{(3)^2}\right) = \frac{1}{9}$$

Since the integral converged to a finite number, our original series will also converge

Note: Series will most likely not converge to 1/9, but it will converge nonetheless.

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Alternating Series Test

- Given a series $\sum (-1)^k a_k$, where a_k is positive for all k
- IF
  - $a_{k+1} \leq a_k$ for all k, and
  - $\lim_{k \to \infty} a_k = 0$

Then the series is convergent

Alternating Series Test Example

Test for convergence

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 4}$$

Check:
Is this series decrease- yes
Is the Lim=0?

$$\lim_{n \to \infty} \frac{n^2}{n^3 + 4} = 0$$

Yes

Therefore, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 4}$, is convergent.

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More Examples

1. $$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}}$$

2. $$\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$$

3. $$\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$$

4. $$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

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Answers

- 1. By Alternating series test, series will converge
- 2. By the comparison test, series will diverge
- 3. By the ratio test, series will converge
- 4. By the integral test, series will diverge